

CONFORMAL WELDINGS AND DISPERSIONLESS TODA HIERARCHY

LEE-PENG TEO

ABSTRACT. Given a C^1 homeomorphism of the unit circle γ , let f and g be respectively the normalized conformal maps from the unit disc and its exterior so that $\gamma = g^{-1} \circ f$ on the unit circle. In this article, we show that by suitably defined time variables, the evolutions of the pairs (g, f) and (g^{-1}, f^{-1}) can be described by an infinite set of nonlinear partial differential equations known as dispersionless Toda hierarchy. Relations to the integrable structure of conformal maps first studied by Wiegmann and Zabrodin [29] are discussed. An extension of the hierarchy which contains both our solution and the solution of [29] is defined.

1. INTRODUCTION

Given a C^1 homeomorphism of the unit circle S^1 , the conformal welding or sewing problem looks for a conformal map f from the unit disc \mathbb{D} and a conformal map g from the exterior of the unit disc \mathbb{D}^* so that $\gamma = g^{-1} \circ f$ on the unit circle. We assume that the origin is contained in $\Omega^+ = f(\mathbb{D})$ and ∞ is contained in $\Omega^- = g(\mathbb{D}^*)$. Under the conditions $f(0) = 0$, $g(\infty) = \infty$, $f'(0)g'(\infty) = 1$, this problem have a unique solution (f, g) .

Starting from the work of Wiegmann and Zabrodin [29], the integrable structure of conformal mappings has aroused considerable interest [17, 24, 20, 6, 13, 16, 14, 15, 2, 3, 18, 22, 4, 5, 1, 31]. It was found that the evolution of conformal mappings can be described by an infinite set of nonlinear partial differential equations, put together under the name dispersionless Toda hierarchy. Later it was revealed that it is also closely related to the Dirichlet boundary problem and two dimensional inverse potential problem [20, 30]. Putting more precisely, Wiegmann and Zabrodin defined a set of time variables on the space of analytic curves. They showed that the deformation of the conformal mapping g of the exterior domain of an analytic curve with respect to these time variables satisfies the dispersionless limit of 2D Toda chain hierarchy. They also defined the notion of tau function for analytic curves, which is the tau function for the hierarchy. By a suitable modification, it was realized that the deformation of the interior conformal mapping f can also be described by the dispersionless Toda hierarchy [20]. However, in their approach, the interior mapping f and exterior mapping

Key words and phrases. conformal welding, dispersionless Toda hierarchy, C^1 homeomorphisms, tau function, Grunsky coefficients.

g are treated separately, i.e. using different time coordinates. Dispersionless Toda hierarchy [27] can be interpreted as describing the evolution of the coefficients of two formal power series $(\mathcal{L}, \tilde{\mathcal{L}})$ with respect to a set of formal time variables t_n , $n \in \mathbb{Z}$. Here $\mathcal{L}(p) = p +$ (lower power terms), and $\tilde{\mathcal{L}}(p) = p +$ higher power terms. Hence under certain analytic conditions, \mathcal{L} is a function univalent in a neighborhood of ∞ , and $\tilde{\mathcal{L}}$ is a function univalent in a neighborhood of the origin. In the work of Wiegmann and Zabrodin, \mathcal{L} is the power series of g and $\tilde{\mathcal{L}}$ is the power series of $1/\bar{g}(z^{-1})$.

In [24], Takhtajan put the work of Wiegmann and Zabrodin in the flavor of conformal field theory. He gave proofs to results in [29, 20, 17] using the idea of sewing or conformal welding. He introduced the Schottky double $\mathbb{P}_{\mathcal{C}}^1$ of the Riemann surface Ω^- — the union of the Riemann surface Ω^- and its complex conjugate copy $\overline{\Omega^-}$ gluing together along their common boundary \mathcal{C} . In this picture $\overline{\Omega^-}$ is the interior of $\mathbb{P}_{\mathcal{C}}^1$ and $1/\bar{g}(z^{-1})$ is its Riemann mapping. Hence we can say that the solution of Wiegmann and Zabrodin [29] describes the sewing problem of $\mathbb{P}_{\mathcal{C}}^1$, where the interior and exterior domains are complex conjugates of each other.

In this paper, we study the integrable structure of the conformal welding problem we discussed at the beginning of the introduction. Let $\text{Homeo}_C(S^1)$ denotes the space of C^1 homeomorphisms of the unit circle. Every C^1 homeomorphism γ has a unique factorization as $g^{-1} \circ f|_{S^1}$, where g is a univalent function on the exterior of the unit disc \mathbb{D}^* , and f is a univalent function on the unit disc \mathbb{D} , subject to the normalization conditions $f(0) = 0$, $g(\infty) = \infty$ and $f'(0)g'(\infty) = 1$. f and g can be extended to be C^1 homeomorphisms of the plane. Let $\gamma(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n e^{i(n+1)\theta}$ be the Fourier expansion of γ . We define t_n by taking half a set of the Fourier coefficients of γ and half a set of the corresponding Fourier coefficients of $1/\gamma^{-1}$. The other half sets of Fourier coefficients of γ and $1/\gamma^{-1}$ are defined as v_n — differentiable functions of t_n on $\text{Homeo}_C(S^1)$. We prove in Proposition 4.4 that the variation of v_m with respect to t_n is equal to $-|mn|b_{n,m}$, when $m \neq 0, n \neq 0$, and $|m|b_{m,0}$ if $m \neq 0, n = 0$, and $-2b_{0,0}$ when $m = n = 0$. Here $b_{n,m}$ are the generalized Grunsky coefficients of the pair of functions (f, g) . A tau function τ is constructed and it is proved that the variation of $\log \tau$ with respect to t_n is equal to v_n . Combining with Proposition 4.4, we get

$$\begin{aligned} \frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_m} &= -|nm|b_{n,m}, \quad n \neq 0, m \neq 0, \\ \frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_0} &= |n|b_{n,0}, \quad n \neq 0, & \frac{\partial^2 \mathcal{F}}{\partial t_0^2} &= -2b_{0,0}, \end{aligned}$$

where $\mathcal{F} = \log \tau$ is called the free energy. Using our result in [25], we conclude that the evolution of the coefficients of the inverse mappings g^{-1} and f^{-1} with respect to t_n satisfies the dispersionless Toda hierarchy. Namely, $(\mathcal{L} = g^{-1}, \tilde{\mathcal{L}} = f^{-1})$ is a solution of the dispersionless Toda hierarchy.

For the pair of mappings (g, f) , we define t_n and v_n as analogues of the t_n and v_n for the inverse mappings (g^{-1}, f^{-1}) , replacing f by f^{-1} and g by g^{-1} . In this case we prove that the variation of v_m with respect to t_n is given by the generalized Grunsky coefficients of the pair (f^{-1}, g^{-1}) . We also construct a tau function τ such that the variation of $\log \tau$ with respect to t_n is equal to v_n . Hence we also conclude that $(\mathcal{L} = g, \tilde{\mathcal{L}} = f)$ satisfies the dispersionless Toda hierarchy.

In the formal level, our solutions to the dispersionless Toda hierarchy, as well as the solution of Wiegmann and Zabrodin, can be viewed as the same solution but with t_n interpreted differently. More precisely, although the v_n 's and t_n 's are defined differently in each of these solutions, considering v_m as functions of the formal variables t_n , they are actually defined by the same functions. Heuristically, this can be seen from the fact that the solutions $(\mathcal{L}, \tilde{\mathcal{L}})$ in these three problems all satisfy the Riemann Hilbert data

$$\mathcal{L}\mathcal{M}^{-1} = \tilde{\mathcal{L}}, \quad \mathcal{M} = \tilde{\mathcal{M}}.$$

Here \mathcal{M} and $\tilde{\mathcal{M}}$ are the Orlov-Schulman functions. The equivalent form of this Riemann Hilbert problem has been studied by Takasaki [23] in connection to string theory. It implies the following string equation

$$\{\mathcal{L}, \tilde{\mathcal{L}}^{-1}\} = 1,$$

which characterize our choice of t_n variables over other choices.

Our solution to the mapping problem and the solution of Wiegmann and Zabrodin can be viewed as two different phases of the same problem if we consider a general problem. Let \mathfrak{D} be the space consists of all pairs of univalent functions (f, g) satisfying the conditions $f(0) = 0$, $g(\infty) = \infty$, $f'(0)g'(\infty) = 1$, f and g can be extended as C^1 homeomorphisms of the plane. We define the functions t_n and v_n , $n \in \mathbb{Z}$ as a direct generalization of those defined for (f, g) associated to a $\gamma \in \text{Homeo}_C(S^1)$. $\{t_n, n \in \mathbb{Z}\}$ can be considered as complex coordinates giving the space \mathfrak{D} a complex structure such that the coefficients of f and g are all analytic functions of t_n . We show that the variations of (g, f) with respect to the variables t_n , $n \in \mathbb{Z}$, satisfy the dispersionless Toda hierarchy. The subspace of \mathfrak{D} defined by the condition $f(z) = 1/\bar{g}(z^{-1})$ is the space considered in the solution of Wiegmann and Zabrodin. The subspace of \mathfrak{D} defined by the condition $f(S^1) = g(S^1)$ corresponds to the space we consider for the conformal welding problem.

The content of the paper is the following. In Section 2 we summarize some facts we need about conformal weldings and univalent functions. In Section 3 we quickly review the dispersionless Toda hierarchy. In Section 4, we define an infinite set of variables t_n and show that the evolution of the pair of inverse mappings (g^{-1}, f^{-1}) satisfies the dispersionless Toda hierarchy. In Section 5, an analogous result is proved for the pair of mappings (g, f) . In Section 6, we discuss the Riemann Hilbert data of our solutions and compare to the solution of Wiegmann and Zabrodin. In Section 7, we discuss a general problem that contains our solution and the solution of [29] as

special cases. In the Appendix, we give an example — the class of conformal mappings to the discs, which corresponds to the subgroup of linear fractional transformations on the unit circle.

2. CONFORMAL WELDINGS AND UNIVALENT FUNCTIONS

We review some concepts we need about conformal weldings and univalent functions. For details, see [11, 12, 19, 9, 21, 8, 25, 26].

Let $\text{Homeo}_C(S^1)$ be the space of all C^1 homeomorphisms on the unit circle S^1 . We denote by \mathbb{D} the unit disc and \mathbb{D}^* the exterior of the unit disc. A C^1 homeomorphism $\gamma \in \text{Homeo}_C(S^1)$ has the following properties:

I. It can be extended to be a C^1 map on the complex plane, which is also denoted by γ , and satisfies

$$\gamma\left(\frac{1}{\bar{z}}\right) = \frac{1}{\gamma(z)}, \quad \forall z \in \mathbb{C}.$$

Moreover, γ is real analytic on $\mathbb{C} \setminus S^1$.

II. There exists two C^1 homeomorphisms f and g of the plane, such that $\gamma = g^{-1} \circ f$, and $f|_{\mathbb{D}}$ and $g|_{\mathbb{D}^*}$ are the unique univalent functions satisfying $f(0) = 0$, $g(\infty) = \infty$ and $f'(0)g'(\infty) = 1$. The decomposition of γ as $g^{-1} \circ f$ is known as conformal welding or sewing. We can associate to γ the simply connected domain $\Omega^+ = f(\mathbb{D}) = g(\mathbb{D})$, its exterior Ω^- and their common boundary $\mathcal{C} = f(S^1) = g(S^1)$, a C^1 curve. Such an association is not one-to-one.

Let $\mathfrak{F}(z) = a_1 z + a_2 z^2 + \dots$ be a function univalent in a neighborhood of the origin and $\mathfrak{G}(z) = bz + b_0 + b_1 z^{-1} + \dots$, $b = a_1^{-1}$ be a function univalent in a neighborhood of ∞ . We define the generalized Grunsky coefficients $b_{m,n}$, $m, n \in \mathbb{Z}$ and Faber polynomials P_n and Q_n by the following formal power series expansion:

$$\log \frac{\mathfrak{G}(z) - \mathfrak{G}(\zeta)}{z - \zeta} = \log b - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} z^{-m} \zeta^{-n},$$

$$\log \frac{\mathfrak{G}(z) - \mathfrak{F}(\zeta)}{z} = \log b - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{m,-n} z^{-m} \zeta^n,$$

$$\log \frac{\mathfrak{F}(z) - \mathfrak{F}(\zeta)}{z - \zeta} = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{-m,-n} z^m \zeta^n,$$

$$\log \frac{\mathfrak{G}(z) - w}{bz} = - \sum_{n=1}^{\infty} \frac{P_n(w)}{n} z^{-n},$$

$$\log \frac{w - \mathfrak{F}(z)}{w} = \log \frac{\mathfrak{F}(z)}{a_1 z} - \sum_{n=1}^{\infty} \frac{Q_n(w)}{n} z^n;$$

and for $m \geq 0$, $n \geq 1$, $b_{-m,n} = b_{n,m}$. By definition, the Grunsky coefficients are symmetric, i.e., $b_{m,n} = b_{n,m}$ for all $m, n \in \mathbb{Z}$. $P_n(w)$ is a polynomial of degree n in w and $Q_n(w)$ is a polynomial of degree n in $1/w$. More precisely,

$$P_n(w) = (\mathfrak{G}^{-1}(w))_{\geq 0}^n, \quad Q_n(w) = (\mathfrak{F}^{-1}(w))_{\leq 0}^{-n}.$$

Here when S is a subset of integers and $A(w) = \sum_n A_n w^n$ is a (formal) power series, we denote by $(A(w))_S = \sum_{n \in S} A_n w^n$.

The functions $\log(\mathfrak{G}(z)/z)$, $P \circ \mathfrak{G}$ and $Q \circ \mathfrak{G}$ are meromorphic in a neighborhood of ∞ and the functions $\log(\mathfrak{F}(z)/z)$, $P_n \circ \mathfrak{F}$ and $Q_n \circ \mathfrak{F}$ are meromorphic in a neighborhood of the origin. Their power series expansions are given by

$$(2.1) \quad \begin{aligned} \log \frac{\mathfrak{G}(z)}{z} &= \log b - \sum_{m=1}^{\infty} b_{m,0} z^{-m}, & \log \frac{\mathfrak{F}(z)}{z} &= \log a_1 - \sum_{m=1}^{\infty} b_{-m,0} z^m \\ P_n(\mathfrak{G}(\zeta)) &= \zeta^n + n \sum_{m=1}^{\infty} b_{nm} \zeta^{-m}, & P_n(\mathfrak{F}(\zeta)) &= n b_{n,0} + n \sum_{m=1}^{\infty} b_{n,-m} \zeta^m, \\ Q_n(\mathfrak{G}(\zeta)) &= -n b_{-n,0} + n \sum_{m=1}^{\infty} b_{m,-n} \zeta^{-m}, & Q_n(\mathfrak{F}(\zeta)) &= \zeta^{-n} + n \sum_{m=1}^{\infty} b_{-n,-m} \zeta^m. \end{aligned}$$

If $(\mathfrak{F}, \mathfrak{G})$ are univalent on \mathbb{D} and \mathbb{D}^* respectively, we also have the following expansions that converge on \mathbb{D} and \mathbb{D}^* respectively:

$$(2.2) \quad \begin{aligned} \frac{1}{\mathfrak{F}(z) - w} &= -\frac{1}{w} + \sum_{n=1}^{\infty} \frac{Q'_n(w)}{n} z^n, & z \in \mathbb{D}, w \in \Omega^- \\ \frac{\mathfrak{F}'(\zeta)}{\mathfrak{F}(\zeta) - \mathfrak{F}(z)} &= \frac{1}{\zeta - z} - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} n b_{-m,-n} z^m \zeta^{n-1}, & z, \zeta \in \mathbb{D}, \\ \frac{1}{\mathfrak{G}(z) - w} &= \sum_{n=1}^{\infty} \frac{P'_n(w)}{n} z^{-n}, & z \in \mathbb{D}^*, w \in \Omega^+, \\ \frac{\mathfrak{G}'(\zeta)}{\mathfrak{G}(\zeta) - \mathfrak{G}(z)} &= \frac{1}{\zeta - z} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m b_{n,m} z^{-n} \zeta^{-m-1}, & z, \zeta \in \mathbb{D}^*. \end{aligned}$$

This follows immediately from the definition. On the other hand, when \mathfrak{F} and \mathfrak{G} are complimentary univalent functions on \mathbb{D} and \mathbb{D}^* , i.e. the exterior of the domain $\mathfrak{F}(\mathbb{D})$ is the domain $\mathfrak{G}(\mathbb{D}^*)$, we proved in [28] that the semi-infinite matrix \mathfrak{B} and \mathfrak{C} defined by

$$\mathfrak{B}_{mn} = \sqrt{nm} b_{-m,n}, \quad \mathfrak{C}_{mn} = \sqrt{mn} b_{m,-n}, \quad m, n \geq 1$$

are invertible matrices. It follows immediately that

Lemma 2.1. *Let \mathfrak{F} and \mathfrak{G} be complimentary univalent functions on \mathbb{D} and \mathbb{D}^* and let P_n, Q_n be their Faber polynomials.*

I. If there exist constants α_n , $n \geq 1$ such that in a neighborhood of the origin,

$$\sum_{n=1}^{\infty} \alpha_n P'_n(z) = 0,$$

then $\alpha_n = 0$ for all n .

II. If there exist constants β_n , $n \geq 1$ such that in a neighborhood of ∞ ,

$$\sum_{n=1}^{\infty} \beta_n Q'_n(z) = 0,$$

then $\beta_n = 0$ for all n .

Proof. In a neighborhood of the origin, we have

$$\sum_{n=1}^{\infty} \alpha_n P'_n(\mathfrak{F}(z)) \mathfrak{F}'(z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nm \alpha_n b_{n,-m} z^{m-1} = \sum_{m=1}^{\infty} \sqrt{m} (\mathfrak{B}\alpha)_m z^{m-1},$$

where $\alpha = (\alpha_1, \sqrt{2}\alpha_2, \dots, \sqrt{n}\alpha_n, \dots)^T$. If $\sum_{n=1}^{\infty} \alpha_n P'_n(z) = 0$, then by uniqueness of power series expansion, we have $\mathfrak{B}\alpha = 0$. By invertibility of \mathfrak{B} , we conclude that $\alpha = 0$. This prove statement I. Statement II is proved similarly. \square

3. DISPERSIONLESS TODA HIERARCHY

The dispersionless Toda hierarchy is a hierarchy of equations describing the evolution of the coefficients of a pair of formal power series $(\mathcal{L}, \tilde{\mathcal{L}})$:

$$(3.1) \quad \begin{aligned} \mathcal{L}(p) &= r(\mathbf{t})p + \sum_{n=0}^{\infty} u_{n+1}(\mathbf{t})p^{-n}, \\ (\tilde{\mathcal{L}}(p))^{-1} &= r(\mathbf{t})p^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(\mathbf{t})p^n. \end{aligned}$$

Here $r(\mathbf{t})$, $u_n(\mathbf{t})$ are functions of t_n , $n \in \mathbb{Z}$, which we denote collectively by \mathbf{t} ; p is a formal variable independent of \mathbf{t} . The evolution of the coefficients u_n are encoded in the following Lax equations:

$$(3.2) \quad \begin{aligned} \frac{\partial \mathcal{L}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{L}\}_T, & \frac{\partial \mathcal{L}}{\partial t_{-n}} &= \{\tilde{B}_n, \mathcal{L}\}_T, \\ \frac{\partial \tilde{\mathcal{L}}}{\partial t_n} &= \{\mathcal{B}_n, \tilde{\mathcal{L}}\}_T, & \frac{\partial \tilde{\mathcal{L}}}{\partial t_{-n}} &= \{\tilde{B}_n, \tilde{\mathcal{L}}\}_T. \end{aligned}$$

Here $\{\cdot, \cdot\}_T$ is the Poisson bracket

$$\{f, g\}_T = p \frac{\partial f}{\partial p} \frac{\partial g}{\partial t_0} - p \frac{\partial f}{\partial t_0} \frac{\partial g}{\partial p}$$

and

$$\mathcal{B}_n = (\mathcal{L}^n)_{>0} + \frac{1}{2}(\mathcal{L}^n)_0, \quad \tilde{B}_n = (\tilde{\mathcal{L}}^n)_{<0} + \frac{1}{2}(\tilde{\mathcal{L}}^n)_0.$$

Proposition 3.1 in [25] can be reformulated as

Proposition 3.1. *If there exists a function \mathcal{F} of \mathbf{t} such that it generates the generalized Grunsky coefficients of a pair $(\mathfrak{F}, \mathfrak{G})$ of formal power series, namely*

$$\frac{\partial^2 \mathcal{F}(\mathbf{t})}{\partial t_m \partial t_n} = \begin{cases} -|mn|b_{m,n}(\mathbf{t}), & \text{if } m \neq 0, n \neq 0 \\ |m|b_{m,0}(\mathbf{t}), & \text{if } m \neq 0, n = 0 \\ -2b_{0,0}(\mathbf{t}), & \text{if } m = n = 0, \end{cases}$$

then the pair of formal power series $(\mathfrak{G}^{-1}, \mathfrak{F}^{-1})$ satisfies the dispersionless Toda hierarchy. Here \mathfrak{G}^{-1} and \mathfrak{F}^{-1} are the inverse functions of \mathfrak{G} and \mathfrak{F} respectively.

4. THE INVERSE MAPPINGS IN CONFORMAL WELDING PROBLEM AND DISPERSIONLESS TODA HIERARCHY

Let $\gamma = g^{-1} \circ f$ be the conformal welding associated to a point $\gamma \in \text{Homeo}_C(S^1)$ and let F and G be the inverse functions of f and g respectively. In this section, we are going to define a set of variables t_n , $n \in \mathbb{Z}$ on the space $\text{Homeo}_C(S^1)$. We prove that the evolution of the mappings (G, F) with respect to these variables satisfies the dispersionless Toda hierarchy.

Given $\gamma = g^{-1} \circ f \in \text{Homeo}_C(S^1)$, we denote by Ω^+ the domain $f(\mathbb{D}) = g(\mathbb{D})$, Ω^- its exterior and $\mathcal{C} = f(S^1) = g(S^1)$ the corresponding C^1 -curve. Let the power series expansion of $f|_{\mathbb{D}}$ and $g|_{\mathbb{D}^*}$ be given by

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} a_n z^n = a_1 z + a_2 z^2 + \dots, \\ g(z) &= bz + \sum_{n=0}^{\infty} b_n z^{-n} = bz + b_0 + b_1 z^{-1} + \dots \end{aligned}$$

They converge on $\mathbb{D} \cup S^1$ and $\mathbb{D}^* \cup S^1$ respectively.

4.1. The variables t_n and v_n . Since γ is C^1 on the unit circle, it has Fourier series expansion on S^1 which converges absolutely. We introduce t_n, v_n , $n < 0$, and t_0 as coefficients of its Fourier series expansion:

$$(4.1) \quad \gamma(w) = \sum_{n=1}^{\infty} -nt_{-n}w^{-n+1} + t_0w + \sum_{n=1}^{\infty} -v_{-n}w^{n+1}, \quad w = e^{i\theta}.$$

Next, we consider the mapping $1/\gamma^{-1} = (1/f^{-1}) \circ g = (1/F) \circ g$. We introduce t_n, v_n , $n > 0$ as coefficients of its Fourier series expansion on S^1 :

$$(4.2) \quad \frac{1}{\gamma^{-1}(w)} = \sum_{n=1}^{\infty} nt_n w^{n-1} + c_0 w^{-1} + \sum_{n=1}^{\infty} v_n w^{-n-1}, \quad w = e^{i\theta}.$$

For the coefficient c_0 , we have

$$c_0 = \frac{1}{2\pi i} \oint_{S^1} \frac{1}{\gamma^{-1}(w)} dw = \frac{1}{2\pi i} \oint_{S^1} \frac{1}{w} d\gamma(w) = \frac{1}{2\pi i} \oint_{S^1} \frac{\gamma(w)}{w^2} dw = t_0.$$

Finally, we define the function v_0 on $\text{Homeo}_C(S^1)$ as

$$v_0 = \frac{1}{2\pi i} \oint_{S^1} \left(\left(\log \frac{f(w)}{w} \right) \frac{\gamma(w)}{w^2} - \left(\log \frac{g(w)}{w} \right) \frac{1}{\gamma^{-1}(w)} \right) dw - \frac{1}{2\pi i} \oint_C \frac{G(z)}{F(z)} \frac{dz}{z}.$$

Remark 4.1. Heuristically, we have

$$v_0 = -\frac{1}{2\pi i} \oint_{S^1} \left((\log w) \frac{\gamma(w)}{w^2} - \frac{\log w}{\gamma^{-1}(w)} \right) dw$$

since heuristically

$$\begin{aligned} \frac{1}{2\pi i} \oint_{S^1} \left((\log f(w)) \frac{\gamma(w)}{w^2} - (\log g(w)) \frac{1}{\gamma^{-1}(w)} \right) dw - \frac{1}{2\pi i} \oint_C \frac{G(z)}{F(z)} \frac{dz}{z} \\ = - \oint_C d \left(\frac{G(z)}{F(z)} \log z \right) = 0, \end{aligned}$$

if we ignore the multi-valued-ness of log function.

To see that $t_n, n \in \mathbb{Z}$ give a complete set of local coordinates for $\text{Homeo}_C(S^1)$, we need the following lemma.

Lemma 4.2. *Let $\gamma_t = g_t^{-1} \circ f_t$ be any deformation of C^1 mappings, then*

$$\frac{(\partial \gamma_t / \partial t) \circ F_t(z)}{F_t(z)^2} F'_t(z) = \left(\frac{\partial}{\partial t} \frac{1}{\gamma_t^{-1}} \right) \circ G_t(z) G'_t(z), \quad z \in C.$$

Proof. Differentiating $\gamma_t \circ \gamma_t^{-1} = \text{id}$ with respect to t and w , we have by chain rule:

$$\begin{aligned} \frac{\partial \gamma_t}{\partial t} \circ \gamma_t^{-1}(w) + \frac{\partial \gamma_t}{\partial w} \circ \gamma_t^{-1}(w) \frac{\partial}{\partial t} (\gamma_t^{-1})(w) &= 0, \\ \frac{\partial \gamma_t}{\partial w} \circ \gamma_t^{-1}(w) \frac{\partial \gamma_t^{-1}(w)}{\partial w} &= 1, \quad w \in S^1. \end{aligned}$$

The assertion follows by setting $w = G(z)$ and using $\gamma_t^{-1} = F_t \circ g_t$. \square

Proposition 4.3. *If $\gamma_t, t \in (-\varepsilon, \varepsilon) \subseteq \mathbb{R}$ is a curve on $\text{Homeo}_C(S^1)$ such that $dt_n/dt = 0$ for all n , then $\gamma_t = \gamma_0$ for all t .*

Proof. If $dt_n/dt|_{t=0} = 0$ for all n , then (4.1) and (4.2) give

$$\frac{d\gamma_t}{dt}(w) = - \sum_{n=1}^{\infty} \frac{dv_{-n}}{dt} w^{n+1}, \quad \left(\frac{d}{dt} \left(\frac{1}{\gamma_t^{-1}} \right) \right) (w) = \sum_{n=1}^{\infty} \frac{dv_n}{dt} w^{-n-1} \quad w \in S^1.$$

Hence $w^{-2}(d\gamma_t/dt)(w)$ is the boundary value of the holomorphic function $-\sum_{n=1}^{\infty} (dv_{-n}/dt) z^{n-1}$ on \mathbb{D} . Since F is holomorphic on Ω^+ ,

$$\frac{(d\gamma^t/dt) \circ F(z)}{F(z)^2} F'(z)$$

is the boundary value of the holomorphic function

$$- \sum_{n=1}^{\infty} \frac{dv_{-n}}{dt} F(z)^{n-1} F'(z)$$

on Ω^+ . On the other hand, $(d(1/\gamma^{-1})/dt)(w)$ is the boundary value of the holomorphic function $\sum_{n=1}^{\infty} (dv_n/dt)z^{-n-1}$ on \mathbb{D}^* . Since G is holomorphic on Ω^- ,

$$\left(\frac{d}{dt}\frac{1}{(\gamma^t)^{-1}}\right) \circ G(z)G'(z)$$

is the boundary value of the holomorphic function

$$\sum_{n=1}^{\infty} \frac{dv_n}{dt} G(z)^{-n-1} G'(z)$$

on Ω^- . Lemma 4.2 then implies that we have a holomorphic function on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which vanishes at ∞ . Hence it must be identically zero. Consequently, $d\gamma_t/dt = 0$ and the assertion follows. \square

4.2. The variations of the functions v_m with respect to t_n .

Proposition 4.4. *Let $b_{m,n}$ be the generalized Grunsky coefficients of the pair of univalent functions $(f|_{\mathbb{D}}, g|_{\mathbb{D}^*})$. The variation of v_m , $m \in \mathbb{Z}, m \neq 0$ with respect to t_n , $n \in \mathbb{Z}$ is given by the following:*

$$\frac{\partial v_m}{\partial t_n} = -|mn|b_{n,m}, \quad n \neq 0, \quad \text{and} \quad \frac{\partial v_m}{\partial t_0} = |m|b_{0,m}.$$

Proof. We follow almost the same idea as the proof of Proposition 4.3. For $n \geq 1$, we have

$$\begin{aligned} \frac{\partial \gamma}{\partial t_n}(w) &= - \sum_{m=1}^{\infty} \frac{\partial v_{-m}}{\partial t_n} w^{m+1}, \\ \frac{\partial}{\partial t_n} \left(\frac{1}{\gamma^{-1}} \right) (w) &= nw^{n-1} + \sum_{m=1}^{\infty} \frac{\partial v_m}{\partial t_n} w^{-m-1}, \quad w \in S^1. \end{aligned}$$

Hence restricted to \mathcal{C} ,

$$\frac{(\partial \gamma / \partial t_n) \circ F(z)}{F(z)^2} F'(z)$$

is the boundary value of the holomorphic function

$$- \sum_{m=1}^{\infty} \frac{\partial v_{-m}}{\partial t_n} F(z)^{m-1} F'(z)$$

on Ω^+ , and

$$\frac{\partial}{\partial t_n} \left(\frac{1}{\gamma^{-1}} \right) \circ G(z)G'(z)$$

is the boundary value of the meromorphic function

$$\left(nG(z)^{n-1} + \sum_{m=1}^{\infty} \frac{\partial v_m}{\partial t_n} G(z)^{-m-1} \right) G'(z)$$

on Ω^- , which has a pole of order $n - 1$ at ∞ . Lemma 4.2 implies that they combine to define a meromorphic function on the plane, with a single pole of order $n - 1$ at ∞ . Hence it is a polynomial of degree $n - 1$, which we denote by $\sum_{m=1}^n \alpha_{n,m} z^{m-1}$. Therefore,

$$(4.3) \quad - \sum_{m=1}^{\infty} \frac{\partial v_{-m}}{\partial t_n} F(z)^{m-1} F'(z) = \sum_{m=1}^n \alpha_{n,m} z^{m-1}, \quad z \in \Omega^+,$$

$$\left(nG(z)^{n-1} + \sum_{m=1}^{\infty} \frac{\partial v_m}{\partial t_n} G(z)^{-m-1} \right) G'(z) = \sum_{m=1}^n \alpha_{n,m} z^{m-1}, \quad z \in \Omega^-.$$

On Ω^- , the second equation gives

$$(4.4) \quad G(z)^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_n} G(z)^{-m} = \sum_{m=1}^n \frac{\alpha_{n,m}}{m} z^m + \alpha_{n,0},$$

where $\alpha_{n,0}$ is an integration constant. Comparing coefficients, we conclude that

$$\sum_{m=1}^n \frac{\alpha_{n,m}}{m} z^m + \alpha_{n,0} = (G(z)^n)_{\geq 0} = P_n(z),$$

where $P_n(z)$ is the n -th Faber polynomial of g . Let $z = g(\zeta)$ in (4.4), and compare to the series expansion of $P_n(g(\zeta))$ given in (2.1), we conclude that for $m, n \geq 1$,

$$\frac{\partial v_m}{\partial t_n} = -mnb_{nm}.$$

Now the first equation in (4.3) gives for $z \in \Omega^+$,

$$- \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_n} F(z)^m = P_n(z) + \tilde{\alpha}_{n,0},$$

where $\tilde{\alpha}_{n,0}$ is another integration constant. Putting $z = f(\zeta)$ and compare to the series expansion of $P_n(f(\zeta))$ given in (2.1), we conclude that for $m, n \geq 1$,

$$\frac{\partial v_{-m}}{\partial t_n} = -mnb_{n,-m}.$$

Now we consider the differentiation with respect to t_0 . Analogous argument shows that restricted to \mathcal{C} ,

$$\frac{(\partial\gamma/\partial t_0) \circ F(z)}{F(z)^2} F'(z) = \frac{\partial}{\partial t_0} \left(\frac{1}{\gamma^{-1}} \right) \circ G(z) G'(z)$$

is the boundary values of the meromorphic function

$$\left(\frac{1}{F(z)} - \sum_{m=1}^{\infty} \frac{\partial v_{-m}}{\partial t_0} F(z)^{m-1} \right) F'(z)$$

on Ω^+ , with a simple pole at the origin; and of the holomorphic function

$$\left(\frac{1}{G(z)} + \sum_{m=1}^{\infty} \frac{\partial v_m}{\partial t_0} G(z)^{-m-1} \right) G'(z)$$

on Ω^- . They combine to give a meromorphic function on $\hat{\mathbb{C}}$, with a single simple pole at the origin. Hence this function must be a multiple of the function $1/z$. Looking at the z^{-1} term of both functions, we conclude that it is $1/z$. After integration, we have

$$\begin{aligned} \log F(z) - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_0} F(z)^m &= \log z + \alpha_0, \quad z \in \Omega^+, \\ \log G(z) - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_0} G(z)^{-m} &= \log z + \tilde{\alpha}_0, \quad z \in \Omega^-. \end{aligned}$$

Put $z = f(\zeta)$ into the first equation and $z = g(\zeta)$ into the second equation, and compare with (2.1), we find that for $m \neq 0$,

$$\frac{\partial v_m}{\partial t_0} = |m|b_{0,m}.$$

Finally differentiation with respect to t_{-n} , $n \geq 1$ shows that the function which equals to

$$\begin{aligned} \left(-nF(z)^{-n-1} - \sum_{m=1}^{\infty} \frac{\partial v_{-m}}{\partial t_{-n}} F(z)^{m-1} \right) F'(z), \quad & z \in \Omega^+, \\ \left(\sum_{m=1}^{\infty} \frac{\partial v_m}{\partial t_{-n}} G(z)^{-m-1} \right) G'(z), \quad & z \in \Omega^-, \end{aligned}$$

is a meromorphic function on $\hat{\mathbb{C}}$ with a single pole at the origin of order $-n-1$. The second equation shows that this function vanishes to the second order at ∞ . Hence we can write this function as $\sum_{m=1}^n \beta_{n,m} z^{-m-1}$. After integration, we get

$$\begin{aligned} F(z)^{-n} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_{-n}} F(z)^m &= \beta_{n,0} - \sum_{m=1}^n \frac{\beta_{n,m}}{m} z^{-m}, \quad z \in \Omega^+, \\ - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_{-n}} G(z)^{-m} &= \tilde{\beta}_{n,0} - \sum_{m=1}^n \frac{\beta_{n,m}}{m} z^{-m}, \quad z \in \Omega^- \end{aligned}$$

Comparing powers in the first equation, we conclude that $\beta_{n,0} - \sum_{m=1}^n \beta_{n,m} z^{-m}/m = Q_n(z)$, where $Q_n(z)$ is the n -th Faber polynomial of f . Finally comparing with the series expansion of $Q_n(f(\zeta))$ and $Q_n(g(\zeta))$ in (2.1) give the desired result

$$\frac{\partial v_m}{\partial t_{-n}} = -|mn|b_{-n,m},$$

for $n \geq 1$ and all $m \neq 0$. □

To study the variation of v_0 with respect to t_n , $n \in \mathbb{Z}$, we need the following 'calculus formula'. Given γ_t a one parameter family in $\text{Homeo}_C(S^1)$, we denote by \mathcal{C}^t the associated C^1 curves. If $h(z, \bar{z}, t)$ is a C^1 function in a neighborhood of \mathcal{C}^t , then

$$(4.5) \quad \left. \frac{d}{dt} \right|_{t=0} \left(\oint_{\mathcal{C}^t} h(z, \bar{z}, t) dz \right) = \oint_{\mathcal{C}} \frac{dh}{dt}(z, \bar{z}, t) dz + \frac{\partial h}{\partial \bar{z}}(z, \bar{z}, t) \left(\frac{\partial \bar{w}^t}{\partial t} dz - \frac{\partial w^t}{\partial t} d\bar{z} \right) \Big|_{t=0}$$

Here $w^t(z, \bar{z})$ is a family of C^1 functions such that $w^0(z, \bar{z}) = \text{id}$, $w^t(\mathcal{C}^t) = \mathcal{C}$. In the following, we are mostly dealing with function h such that $h_{\bar{z}} = 0$ on \mathcal{C} . In this case the second term is missing.

Proposition 4.5. *The variation of v_0 with respect to t_n , $n \in \mathbb{Z}$ is given by*

$$\frac{\partial v_0}{\partial t_n} = |n|b_{n,0} = \frac{\partial v_n}{\partial t_0}, \quad n \neq 0, \quad \frac{\partial v_0}{\partial t_0} = -2 \log b = 2 \log a_1.$$

Proof. We can check that for any t , the expression

$$\frac{\partial \log f_t}{\partial t} \circ F_t(z) \frac{G_t(z) F'_t(z)}{F_t(z)^2} - \frac{\partial \log g_t}{\partial t} \circ G_t(z) \frac{G'_t(z)}{F_t(z)} - \frac{1}{z} \left(\frac{\partial}{\partial t} \frac{G_t}{F_t} \right) (z),$$

vanishes identically on \mathcal{C}_t . Together with (4.5), we have

$$\frac{\partial v_0}{\partial t_n} = \frac{1}{2\pi i} \oint_{S^1} \left(\log \frac{f(w)}{w} \right) \frac{(\partial \gamma / \partial t_n)(w)}{w^2} - \left(\log \frac{g(w)}{w} \right) \left(\frac{\partial}{\partial t_n} \frac{1}{\gamma^{-1}} \right) (w) dw$$

Using the series expansion for each term give the desired result. \square

Since the Grunsky coefficients are symmetric, from Proposition 4.4 and Proposition 4.5, formally there should exist a function \mathcal{F} on $\text{Homeo}_C S^1$ such that $\partial \mathcal{F} / \partial t_n = v_n$. We define this function in the next section.

4.3. Tau function. First for $\gamma \in \text{Homeo}_C(S^1)$, we define the following two functions.

$$\psi(z) = \sum_{n=1}^{\infty} \frac{v_{-n}}{n} z^n, \quad z \in \mathbb{D}, \quad \phi(z) = \sum_{n=1}^{\infty} \frac{v_n}{n} z^{-n}, \quad z \in \mathbb{D}^*.$$

Since γ and $1/\gamma^{-1}$ are C^1 functions on S^1 , these are holomorphic functions on \mathbb{D} and \mathbb{D}^* respectively. We define the tau function τ by the following formula

$$\begin{aligned} 4 \log \tau &= 2t_0 v_0 - t_0^2 + \frac{1}{2\pi i} \oint_{S^1} \frac{1}{\gamma^{-1}(w)} (w\phi'(w) + 2\phi(w)) dw \\ &\quad + \frac{1}{2\pi i} \oint_{S^1} \frac{\gamma(w)}{w^2} (w\psi'(w) - 2\psi(w)) dw. \end{aligned}$$

Remark 4.6. When the sum converges absolutely, the tau function can be written explicitly as

$$4 \log \tau = 2t_0 v_0 - t_0^2 - \sum_{n=1}^{\infty} (n-2)(t_n v_n + t_{-n} v_{-n}).$$

We can compare this to the explicit formula for the tau function of Wiegmann and Zabrodin [29, 17]. We are going to explain this coincidence in a latter discussion.

We want to prove that the free energy $\mathcal{F} = \log \tau$ generates the variables v_n , $n \in \mathbb{Z}$. First, from Proposition 4.4 and the identities in (2.1), we have the following variational formulas:

Lemma 4.7. *The variations of the functions ψ and ϕ with respect to t_n , t_{-n} , $n \geq 1$ and t_0 are given by*

$$\begin{aligned} \frac{\partial \psi}{\partial t_n}(z) &= -P_n(f(z)) + nb_{n,0}, & \frac{\partial \phi}{\partial t_n}(z) &= -P_n(g(z)) + z^n, \\ \frac{\partial \psi}{\partial t_0}(z) &= -\log \frac{f(z)}{z} + \log a_1, & \frac{\partial \phi}{\partial t_0}(z) &= -\log \frac{g(z)}{z} + \log b, \\ \frac{\partial \psi}{\partial t_{-n}}(z) &= -Q_n(f(z)) + z^{-n}, & \frac{\partial \phi}{\partial t_{-n}}(z) &= -Q_n(g(z)) - nb_{-n,0}. \end{aligned}$$

Now we prove our claim.

Proposition 4.8. *The tau function generates the functions v_n , namely*

$$\frac{\partial \log \tau}{\partial t_n} = v_n.$$

for all $n \in \mathbb{Z}$.

Proof. For $n \geq 1$, we have

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{S^1} \left(\left(\frac{\partial}{\partial t_n} \frac{1}{\gamma^{-1}} \right) (w) (w\phi'(w) + 2\phi(w)) + \frac{(\partial \gamma / \partial t_n)(w)}{w^2} (w\psi'(w) - 2\psi(w)) \right) dw \\ &= -(n-2)v_n, \end{aligned}$$

from their explicit series expansion. On the other hand, from Lemma 4.7, we have

$$\begin{aligned}
& \frac{1}{2\pi i} \oint_{S^1} \frac{1}{\gamma^{-1}(w)} \left(w \frac{\partial}{\partial w} \left(\frac{\partial \phi}{\partial t_n} \right) (w) + 2 \frac{\partial \phi}{\partial t_n} (w) \right) dw \\
& + \frac{1}{2\pi i} \oint_{S^1} \frac{\gamma(w)}{w^2} \left(w \frac{\partial}{\partial w} \left(\frac{\partial \psi}{\partial t_n} \right) (w) - 2 \frac{\partial \psi}{\partial t_n} (w) \right) dw \\
& = (n+2)v_n - 2nt_0 b_{n,0} - \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\frac{G(z)}{F(z)} P'_n(z) + 2 \frac{G'(z)}{F(z)} P_n(z) \right) dz \\
& \quad - \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\frac{G(z)}{F(z)} P'_n(z) - 2 \frac{G(z)F'(z)}{F(z)^2} P_n(z) \right) dz \\
& = (n+2)v_n - 2nt_0 b_{n,0} - \frac{1}{2\pi i} \oint_{\mathcal{C}} d \left(\frac{G(z)}{F(z)} P_n(z) \right) \\
& = (n+2)v_n - 2nt_0 b_{n,0}.
\end{aligned}$$

Together with Proposition 4.5, we have

$$4 \frac{\partial \log \tau}{\partial t_n} = 2nt_0 b_{n,0} - (n-2)v_n + (n+2)v_n - 2nt_0 b_{n,0} = 4v_n.$$

The cases where $n = 0$ and $n \leq -1$ are proved analogously. \square

Combining this proposition with Proposition 4.4 and Proposition 4.5, we have

$$(4.6) \quad \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = \begin{cases} -|mn|b_{m,n}, & \text{if } m \neq 0, n \neq 0 \\ |m|b_{m,0}, & \text{if } m \neq 0, n = 0 \\ -2b_{0,0}, & \text{if } m = n = 0; \end{cases}$$

where $\mathcal{F} = \log \tau$. By Proposition 3.1, we conclude that

Theorem 4.9. *The evolution of the functions (G, F) with respect to t_n satisfies the dispersionless Toda hierarchy (3.2).*

5. THE CONFORMAL MAPPINGS IN CONFORMAL WELDING PROBLEM AND DISPERSIONLESS TODA HIERARCHY

In this section, we want to prove that the evolution of the functions (g, f) in the conformal welding with respect to some suitably defined t_n satisfies the dispersionless Toda hierarchy. The definitions of t_n mimic what we did in the previous section, by replacing f with F and g with G . However, things become more involved since we do not have the unit circle S^1 where we can write down convergent series expansion for the functions involved. Instead we have to work with the curves \mathcal{C} .

5.1. The variables t_n and v_n . Given $\gamma = g^{-1} \circ f \in \text{Homeo}_C(S^1)$, we construct the following differentiable functions on $\text{Homeo}_C(S^1)$. We use the same notation as the previous section since it shall not incur confusion. For $n \geq 1$, we define

$$\begin{aligned} t_n &= \frac{1}{2\pi i n} \oint_{S^1} \frac{(g(w))^{-n}}{f(w)} dg(w) = \frac{1}{2\pi i n} \oint_{\mathcal{C}} \frac{z^{-n}}{f \circ G(z)} dz, \\ t_{-n} &= \frac{-1}{2\pi i n} \oint_{S^1} g(w)(f(w))^{n-2} df(w) = \frac{-1}{2\pi i n} \oint_{\mathcal{C}} g \circ F(z) z^{n-2} dz, \\ v_n &= \frac{1}{2\pi i} \oint_{S^1} \frac{(g(w))^n}{f(w)} dg(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{z^n}{f \circ G(z)} dz, \\ v_{-n} &= \frac{-1}{2\pi i} \oint_{S^1} g(w)(f(w))^{-n-2} df(w) = \frac{-1}{2\pi i} \oint_{\mathcal{C}} g \circ F(z) z^{-n-2} dz; \end{aligned}$$

while for $n = 0$,

$$\begin{aligned} t_0 &= \frac{1}{2\pi i} \oint_{S^1} \frac{1}{f(w)} dg(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{f \circ G(z)} dz = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{g \circ F}{z^2} dz, \\ v_0 &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\left(\log \frac{F(z)}{z} \right) \frac{g \circ F(z)}{z^2} - \left(\log \frac{G(z)}{z} \right) \frac{1}{f \circ G(z)} \right) dz \\ &\quad - \frac{1}{2\pi i} \oint_{S^1} \frac{g(z)}{f(z)} \frac{dz}{z}. \end{aligned}$$

Unlike the previous case where the t_n and v_n appear as Fourier coefficients, we do not understand the significance of the t_n and v_n here.

We also introduce the following functions:

$$(5.1) \quad S_{\pm}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(1/f) \circ G(w)}{w - z} dw, \quad \tilde{S}_{\pm}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{g \circ F(w)}{w^2(w - z)} dw.$$

Here S_+ and \tilde{S}_+ are defined for $z \in \Omega^+$. They are holomorphic functions on Ω^+ . In a neighborhood of the origin, they have the series expansion

$$(5.2) \quad S_+(z) = \sum_{n=1}^{\infty} n t_n z^{n-1}, \quad \tilde{S}_+(z) = - \sum_{n=1}^{\infty} v_{-n} z^{n-1}.$$

S_- and \tilde{S}_- are defined for $z \in \Omega^-$. In a neighborhood of infinity, they have the series expansion

$$(5.3) \quad S_-(z) = -\frac{t_0}{z} - \sum_{n=1}^{\infty} v_n z^{-n-1}, \quad \tilde{S}_-(z) = -\frac{t_0}{z} + \sum_{n=1}^{\infty} n t_{-n} z^{-n-1}.$$

The theory of complex analysis tells us that on the curve \mathcal{C} ,

$$(5.4) \quad (1/f) \circ G(z) = S_+(z) - S_-(z), \quad \frac{g \circ F(z)}{z^2} = \tilde{S}_+(z) - \tilde{S}_-(z).$$

To prove that t_n , $n \in \mathbb{Z}$ give a complete set of local coordinates on $\text{Homeo}_C(S^1)$, we need the analog of Lemma 4.2:

Lemma 5.1. *Given a one-parameter curve $\gamma_t = g_t^{-1} \circ f_t$ on $\text{Homeo}_C(S^1)$, we have*

$$\frac{(\partial(g_t \circ f_t^{-1})/\partial t) \circ f_t(z)}{(f_t(z))^2} f'_t(z) = \left(\frac{\partial}{\partial t} \frac{1}{f_t \circ g_t^{-1}} \right) \circ g_t(z) g'_t(z), \quad z \in S^1.$$

The proof is the same as in Lemma 4.2.

Proposition 5.2. *If γ_t , $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ is a curve on $\text{Homeo}_C(S^1)$ such that $dt_n/dt = 0$ for all t , then $\gamma_t = \gamma_0$ for all t .*

Proof. If γ_t is a one parameter family in $\text{Homeo}_C(S^1)$ such that $dt_n/dt = 0$ for all n , then (5.2) and (5.3) give

$$\frac{dS_+}{dt}(z) = 0, \quad \frac{d\tilde{S}_-}{dt}(z) = 0;$$

and

$$\frac{dS_-}{dt} = O(z^{-2}) \quad \text{as } z \rightarrow \infty.$$

Hence we conclude from (5.4) that $d((1/f) \circ G)/dt$ is the boundary value of the holomorphic function dS_-/dt on Ω^- , which vanish at ∞ . Since g is holomorphic on \mathbb{D}^* ,

$$\frac{d}{dt} \left(\frac{1}{f \circ G} \right) \circ g(z) g'(z), \quad z \in S^1$$

is the boundary value of a holomorphic function on \mathbb{D}^* . Similarly, we conclude that

$$\frac{(d(g \circ F)/dt) \circ f(z)}{(f(z))^2} f'(z), \quad z \in S^1$$

is the boundary value of a holomorphic function on \mathbb{D} . Lemma 5.1 then implies that we have a holomorphic function on $\hat{\mathbb{C}}$ which vanishes at ∞ . This function must vanish identically. Therefore $d(g \circ F)/dt = 0$ and working this formula out explicitly, we have

$$\frac{1}{g'(z)} \frac{dg(z)}{dt} = \frac{1}{f'(z)} \frac{df(z)}{dt}.$$

However, restricted to S^1 ,

$$\begin{aligned} \frac{1}{g'(z)} \frac{dg(z)}{dt} &= \frac{d \log b}{dt} z + \text{lower order terms in } z. \\ \frac{1}{f'(z)} \frac{df(z)}{dt} &= \frac{d \log a_1}{dt} z + \text{higher order terms in } z. \end{aligned}$$

Comparing coefficients and using the fact that $a_1 = b^{-1}$, we conclude that $dg/dt = df/dt = 0$, and consequently $d\gamma_t/dt = 0$. \square

5.2. The variations of the functions v_m with respect to t_n . We still denote by $b_{m,n}$ the generalized Grunsky coefficients of (f, g) , and P_n, Q_n their Faber polynomials. We use $\kappa_{m,n}, \mathcal{P}_n, \mathcal{Q}_n$ to denote the generalized Grunsky coefficients and Faber polynomials of $(F = f^{-1}, G = g^{-1})$.

Proposition 5.3. *For $m \neq 0$, the variation of the function v_m with respect to the variables t_n is given by the following:*

$$\frac{\partial v_m}{\partial t_n} = -|mn|\kappa_{n,m}, \quad n \neq 0, \quad \frac{\partial v_m}{\partial t_0} = |m|\kappa_{0,m}.$$

Proof. First we consider the variation with respect to t_n , $n \geq 1$. For $w \in S^1$, let

$$(5.5) \quad \frac{(\partial(g \circ F)/\partial t_n) \circ f(w)f'(w)}{(f(w))^2} = \frac{\partial}{\partial t_n} \left(\frac{1}{f \circ G} \right) \circ g(w)g'(w) = \sum_{m \in \mathbb{Z}} \alpha_{n,m} w^m.$$

From (5.3), we conclude that $\partial \tilde{S}_- / \partial t_n = 0$. On the other hand, using the definition of \tilde{S}_- , we have on Ω^-

$$0 = \frac{\partial \tilde{S}_-}{\partial t_n} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\partial(g \circ F)(w)/\partial t_n}{w^2(w-z)} dw = \frac{1}{2\pi i} \oint_{S^1} \frac{(\partial(g \circ F)/\partial t_n) \circ f(w)f'(w)}{(f(w))^2(f(w)-z)} dw.$$

Using the expansion of $1/(f(w)-z)$ given by (2.2) to compute this integral, we get

$$0 = -\alpha_{n,-1}z^{-1} + \sum_{m=1}^{\infty} \frac{\alpha_{n,-m-1}}{m} Q'_m(z)$$

for z in a neighborhood of ∞ . Since

$$Q'_m(z) = O(z^{-2}), \quad \text{as } z \rightarrow \infty,$$

we have $\alpha_{n,-1} = 0$. It follows from Lemma 2.1 that $\alpha_{n,m} = 0$ for all $m \leq -1$. For $z \in \Omega^+$, we have by (5.2):

$$nz^{n-1} = \frac{\partial S_+}{\partial t_n}(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{(\partial((1/(f \circ G))/\partial t_n) \circ g(w)g'(w))}{g(w)-z} dw.$$

Using the series expansion of $1/(g(w)-z)$ given by (2.2), we have for z in a neighborhood of the origin,

$$(5.6) \quad nz^{n-1} = \sum_{m=1}^{\infty} \frac{\alpha_{n,m-1}}{m} P'_m(z).$$

On the other hand, using (2.2) and (5.5), we have for $\zeta \in \mathbb{D}^*$,

(5.7)

$$\begin{aligned} \frac{\partial S_-}{\partial t_n} \circ g(\zeta)g'(\zeta) &= \frac{-1}{2\pi i} \oint_{S^1} \frac{\partial}{\partial t_n} \left(\frac{1}{f \circ g^{-1}} \right) \circ g(w)g'(w) \frac{g'(\zeta)}{g(\zeta) - g(w)} dw \\ &= - \sum_{m=1}^{\infty} \alpha_{n,m-1} \sum_{k=1}^{\infty} k b_{km} \zeta^{-k-1} = \sum_{m=1}^{\infty} \frac{\alpha_{n,m-1}}{m} \frac{\partial}{\partial \zeta} (P_m(g(\zeta)) - \zeta^m) \\ &= \frac{\partial}{\partial \zeta} \left(g(\zeta)^n - \sum_{m=1}^{\infty} \frac{\alpha_{n,m-1}}{m} \zeta^m \right). \end{aligned}$$

The last equality follows from (5.6). Since the power series expansion at infinity of the left hand side only have negative power terms, we conclude that

$$(5.8) \quad \sum_{m=1}^{\infty} \frac{\alpha_{n,m-1}}{m} \zeta^m + c_1 = (g(\zeta))_{\geq 0}^n = \mathcal{P}_n(\zeta),$$

where c_1 is an integration constant and \mathcal{P}_n is the n -th Faber polynomial of G . This implies that $\alpha_{n,m} = 0$ for $m \geq n$. Putting $\zeta = G(z)$ in (5.7), integrating, comparing the series expansion at ∞ using (5.3) and (5.8), we have

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_n} z^{-m} = z^n - \mathcal{P}_n(G(z)) + c_1.$$

From this we conclude that for $m, n \geq 1$,

$$\frac{\partial v_m}{\partial t_n} = -mn\kappa_{n,m}.$$

Next, consider the holomorphic function on \mathbb{D}

$$\frac{\partial \tilde{S}_+}{\partial t_n} \circ f(\zeta)f'(\zeta) = \frac{-1}{2\pi i} \oint_{S^1} \frac{(\partial(g \circ f^{-1})/\partial t_n) \circ f(w)f'(w)}{(f(w))^2} \frac{f'(\zeta)}{f(\zeta) - f(w)} dw.$$

Using (2.2), (5.5) and (5.8), we have

$$\frac{\partial \tilde{S}_+}{\partial t_n} \circ f(\zeta)f'(\zeta) = \sum_{m=0}^{n-1} \alpha_{n,m} \zeta^m = \mathcal{P}'_n(\zeta).$$

Putting $\zeta = F(z)$, integrating, comparing the series expansion in a neighborhood of the origin and using (5.2), we find that

$$- \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_n} z^m = \mathcal{P}_n(f^{-1}(z)) + c_2.$$

This implies for $m, n \geq 1$

$$\frac{\partial v_{-m}}{\partial t_n} = -mn\kappa_{n,-m}.$$

The variation with respect to t_{-n} , $n \geq 1$ follows the same idea. Let

$$(5.9) \quad \frac{(\partial(g \circ f^{-1})/\partial t_{-n}) \circ f(w)f'(w)}{(f(w))^2} = \frac{\partial}{\partial t_{-n}} \left(\frac{1}{f \circ g^{-1}} \right) \circ g(w)g'(w) = \sum_{m \in \mathbb{Z}} \beta_{n,m} w^m$$

on S^1 . Since $\partial S_+/\partial t_{-n} = 0$,

$$\sum_{m=1}^{\infty} \frac{\beta_{n,m-1}}{m} P'_m(z) = 0.$$

By Lemma 2.1, this implies $\beta_{n,m} = 0$ for $m \geq 0$. Next since $\partial \tilde{S}_-(z)/\partial t_{-n} = nz^{-n-1}$, we have

$$-\beta_{n,-1}z^{-1} + \sum_{m=1}^{\infty} \frac{\beta_{n,-m-1}}{m} Q'_m(z) = nz^{-n-1}$$

on a neighborhood of ∞ . This implies that $\beta_{n,-1} = 0$. Now for $\zeta \in \mathbb{D}$, we have

$$(5.10) \quad \begin{aligned} \frac{\partial \tilde{S}_+}{\partial t_{-n}} \circ f(\zeta)f'(\zeta) &= \sum_{m=1}^{\infty} \beta_{n,-m-1} \sum_{k=1}^{\infty} kb_{-k,-m} \zeta^{k-1} \\ &= \sum_{m=1}^n \frac{\beta_{n,-m-1}}{m} \frac{\partial}{\partial \zeta} (Q_m(f(\zeta)) - \zeta^{-m}) = \frac{\partial}{\partial \zeta} \left(-(f(\zeta))^{-n} - \sum_{m=1}^{\infty} \frac{\beta_{n,-m-1}}{m} \zeta^{-m} \right). \end{aligned}$$

Since the left hand side does not contain negative powers of ζ , we conclude that $\beta_{n,m} = 0$ for $m \leq -n-2$ and for some constant c_3 ,

$$\sum_{m=1}^n \frac{\beta_{n,-m-1}}{m} \zeta^{-m} + c_3 = -(f(\zeta))_{\leq 0}^{-n} = -\mathcal{Q}_n(\zeta),$$

where \mathcal{Q}_n is the n -th Faber polynomial of F . Together with (5.10), this implies that in a neighborhood of the origin, we have

$$-\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_{-n}} z^m = \mathcal{Q}_n(F(z)) - z^{-n} + c_3.$$

Hence we conclude that for $m, n \geq 1$,

$$\frac{\partial v_{-m}}{\partial t_{-n}} = -mn\kappa_{-n,-m}.$$

Finally, for $\zeta \in \mathbb{D}^*$, we have

$$\frac{\partial S_-}{\partial t_{-n}} \circ g(\zeta)g'(\zeta) = -\sum_{m=1}^n \beta_{n,-m-1} \zeta^{-m-1} = -\mathcal{Q}'_n(\zeta).$$

Hence in a neighborhood of ∞ , we have

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_{-n}} z^{-m} = -\mathcal{Q}_n(G(z)) + c_4.$$

This implies that for $m, n \geq 1$

$$\frac{\partial v_m}{\partial t_{-n}} = -mn\kappa_{-n,m}.$$

For variation with respect to t_0 , let

$$(5.11) \quad \frac{(\partial(g \circ F)/\partial t_0) \circ f(w)f'(w)}{(f(w))^2} = \frac{\partial}{\partial t_0} \left(\frac{1}{f \circ G} \right) \circ g(w)g'(w) = \sum_{m \in \mathbb{Z}} \chi_m w^m$$

on S^1 . Since $\partial S_+(z)/\partial t_0 = 0$, $\chi_m = 0$ for all $m \geq 0$. On the other hand, since $\partial \tilde{S}_-/\partial t_0 = -1/z$, this implies that

$$-\chi_{-1}z^{-1} + \sum_{m=1}^{\infty} \frac{\chi_{-m-1}}{m} Q'_m(z) = -z^{-1}$$

in a neighborhood of infinity. Consequently, $\chi_{-1} = 1$ and by Lemma 2.1, $\chi_m = 0$ for all $m \leq -2$. Now using the formula (5.1) for S_- , we find that

$$\frac{\partial S_-}{\partial t_0} \circ g(\zeta)g'(\zeta) = -\frac{1}{\zeta}, \quad \zeta \in \mathbb{D}^*.$$

Together with (5.3), this implies that in a neighborhood of ∞ , we have

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_0} z^{-m} = -\log \frac{G(z)}{z} - \log b.$$

Eq. (2.1) then shows that for $m > 0$,

$$\frac{\partial v_m}{\partial t_0} = m\kappa_{0,m}.$$

Finally, we have for $\zeta \in \mathbb{D}$,

$$\frac{\partial \tilde{S}_+}{\partial t_0} \circ f(\zeta)f'(\zeta) = \sum_{k=1}^{\infty} kb_{-k,0}\zeta^{k-1} = -\frac{\partial}{\partial \zeta} \left(\log \frac{f(\zeta)}{\zeta} \right).$$

Together with (5.2), this implies that in a neighborhood of the origin

$$-\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_{-m}}{\partial t_0} z^m = \log \frac{F(z)}{z},$$

which gives us

$$\frac{\partial v_{-m}}{\partial t_0} = m\kappa_{0,-m}$$

for $m > 0$. This concludes the proof. □

We gather some facts we need later from the proof above in the following corollary.

Corollary 5.4. *We have the following variational formula for the functions S_{\pm} and \tilde{S}_{\pm} . For $n \geq 1$,*

$$\begin{aligned} \frac{\partial S_+}{\partial t_n}(z) &= nz^{n-1}, & \frac{\partial S_+}{\partial t_{-n}} &= 0, \\ \frac{\partial S_-}{\partial t_n}(z) &= -\frac{\partial}{\partial z}(\mathcal{P}_n(G(z)) - z^n), & \frac{\partial S_-}{\partial t_{-n}}(z) &= -\frac{\partial}{\partial z}(\mathcal{Q}_n(G(z))) \\ \frac{\partial \tilde{S}_+}{\partial t_n}(z) &= \frac{\partial}{\partial z}(\mathcal{P}_n(F(z))), & \frac{\partial \tilde{S}_+}{\partial t_{-n}}(z) &= \frac{\partial}{\partial z}(\mathcal{Q}_n(F(z)) - z^{-n}) \\ \frac{\partial \tilde{S}_-}{\partial t_n}(z) &= 0, & \frac{\partial \tilde{S}_-}{\partial t_{-n}}(z) &= nz^{-n-1} \end{aligned}$$

For $n = 0$,

$$\begin{aligned} \frac{\partial S_+}{\partial t_0}(z) &= 0, & \frac{\partial S_-}{\partial t_0}(z) &= -\frac{G'(z)}{G(z)}, \\ \frac{\partial \tilde{S}_+}{\partial t_0}(z) &= \frac{\partial}{\partial z} \log \frac{F(z)}{z}, & \frac{\partial \tilde{S}_-}{\partial t_0}(z) &= -\frac{1}{z}. \end{aligned}$$

Now we consider the variation of v_0 .

Proposition 5.5. *The variation of v_0 with respect to t_n , $n \in \mathbb{Z}$ is given by*

$$\frac{\partial v_0}{\partial t_n} = |n| \kappa_{n,0}, \quad m \neq 0, \quad \frac{\partial v_0}{\partial t_0} = 2 \log b = -2 \log a_1.$$

Proof. The proof parallels the proof of Proposition 4.5. We have,

$$\begin{aligned} \frac{\partial v_0}{\partial t_n} &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\log \frac{F(z)}{z} \right) \frac{\partial g \circ F(z)}{\partial t_n} z^{-2} dz - \left(\log \frac{G(z)}{z} \right) \left(\frac{\partial}{\partial t_n} \frac{1}{f \circ G} \right) (z) dz \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\log \frac{F(z)}{z} \right) \left(\frac{\partial \tilde{S}_+}{\partial t_n} - \frac{\partial \tilde{S}_-}{\partial t_n} \right) (z) dz \\ &\quad - \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\log \frac{G(z)}{z} \right) \left(\frac{\partial S_+}{\partial t_n} - \frac{\partial S_-}{\partial t_n} \right) (z) dz. \end{aligned}$$

For $n \geq 1$, since $\log(F(z)/z)$ and $\partial \tilde{S}_{\pm}/\partial t_n$ are both holomorphic functions on Ω^+ , Cauchy integral formula implies that the first integral vanishes. Similarly, for the second integral, we are left with

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\log \frac{G(z)}{z} \right) (nz^{n-1}) dz = n \kappa_{n,0}.$$

The cases $n \leq 0$ are proved similarly. \square

Since the Grunsky coefficients are symmetric, Propositions 5.3 and 5.5 implies that there exists a function \mathcal{F} on $\text{Homeo}_C(S^1)$ such that $\partial \mathcal{F}/\partial t_n = v_n$. We are going to define this function in the next section.

5.3. Tau function. First we define the following two functions. In a neighborhood of ∞ , they are given by

$$\Psi(z) = \sum_{n=1}^{\infty} \frac{v_{-n}}{n} z^n, \quad \Phi(z) = \sum_{n=1}^{\infty} \frac{v_n}{n} z^{-n}.$$

Since $\Psi'(z) = -\tilde{S}_+(z)$, $\Psi(z)$ can be analytically continued to a holomorphic function on Ω^+ . Analogously, $\Phi'(z) = S_-(z) + t_0/z$ implies that $\Phi(z)$ can be analytically continued to be a holomorphic function on Ω^- .

The tau function is defined in the similar way as in Section 4:

$$\begin{aligned} 4 \log \tau = & 2t_0 v_0 - t_0^2 + \frac{1}{2\pi i} \oint_C \frac{1}{f \circ G(z)} (z\Phi'(z) + 2\Phi(z)) dz \\ & + \frac{1}{2\pi i} \oint_C \frac{g \circ F(z)}{z^2} (z\Psi'(z) - 2\Psi(z)) dz. \end{aligned}$$

We use the same notation τ for the tau function since it will not incur any confusion. We are going to show that $\mathcal{F} = \log \tau$ generates the functions v_n .

First from Corollary 5.4, we have

Lemma 5.6. *The variations of the functions Φ and Ψ are given by*

$$\begin{aligned} \frac{\partial \Psi}{\partial t_n}(z) &= -\mathcal{P}_n(F(z)) + n\kappa_{n,0}, & \frac{\partial \Phi}{\partial t_n}(z) &= -\mathcal{P}_n(G(z)) + z^n, \\ \frac{\partial \Psi}{\partial t_0}(z) &= \log \frac{F(z)}{z} + \log a_1, & \frac{\partial \Phi}{\partial t_0}(z) &= -\log \frac{G(z)}{z} - \log b, \\ \frac{\partial \Psi}{\partial t_{-n}}(z) &= -\mathcal{Q}_n(F(z)) + z^{-n}, & \frac{\partial \Phi}{\partial t_{-n}}(z) &= -\mathcal{Q}_n(G(z)) - n\kappa_{-n,0} \end{aligned}$$

The integration constants are found by comparing power series expansion of both sides in a neighborhood of the origin or ∞ .

Now we can state our proposition.

Proposition 5.7. *The tau function generates the functions v_n on $\text{Homeo}_C(S^1)$, namely*

$$\frac{\partial \log \tau}{\partial t_n} = v_n,$$

for all $n \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Proposition 4.8. Instead of using series expansion on S^1 , we use the Cauchy integral formula and residue calculus as we do in the proof of Proposition 5.5. \square

From this, we conclude as Theorem 4.9 that

Theorem 5.8. *The evolution of the conformal mappings (g, f) with respect to the variables t_n satisfies the dispersionless Toda hierarchy.*

5.4. Symmetry of the variables under $z \mapsto 1/z$ transformation. We study the symmetries of the coordinates t_n . The transformation $z \mapsto 1/z$ on the complex plane will interchange the interior and exterior. The domain Ω^+ that contains the origin will be mapped to the domain $1/\Omega^+$ that contains ∞ . If f and g are the Riemann mappings of Ω^+ and Ω^- such that $f'(0)g'(\infty) = 1$, then $\tilde{f}(z) = 1/g(z^{-1})$ and $\tilde{g}(z) = 1/f(z^{-1})$ are the Riemann mappings of the domains $1/\Omega^-$ and $1/\Omega^+$, and $\tilde{f}'(0)\tilde{g}'(\infty) = 1$. Observe that

$$\begin{aligned} t_n &= \frac{1}{2\pi i n} \oint_{S^1} \frac{g(w)^{-n}}{f(w)} dg(w) = \frac{1}{2\pi i n} \oint_{S^1} \frac{(g(w^{-1}))^{-n}}{f(w^{-1})} dg(w^{-1}) \\ &= \frac{-1}{2\pi i n} \oint_{S^1} (\tilde{f}(w))^{n-2} \tilde{g}(w) d\tilde{f}(w) \end{aligned}$$

and similarly

$$\begin{aligned} t_{-n} &= \frac{1}{2\pi i n} \oint_{S^1} \frac{(\tilde{g}(w))^{-n}}{\tilde{f}(w)} d\tilde{g}(w), \\ v_n &= \frac{-1}{2\pi i} \oint_{S^1} (\tilde{f}(w))^{-n-2} \tilde{g}(w) d\tilde{f}(w), \quad v_{-n} = \frac{1}{2\pi i} \oint_{S^1} \frac{(\tilde{g}(w))^n}{\tilde{f}(w)} d\tilde{g}(w). \end{aligned}$$

This agrees with our observation in [25] that if $(\mathcal{L}, \tilde{\mathcal{L}})$ is a solution of the dispersionless Toda hierarchy, then $(\mathcal{L}', \tilde{\mathcal{L}}')$, where $\mathcal{L}'(p) = 1/\tilde{\mathcal{L}}(p^{-1})$, $\tilde{\mathcal{L}}'(p) = 1/\mathcal{L}(p^{-1})$, is also a solution of the dispersionless Toda hierarchy, when we define the new time variables as $t'_n = t_{-n}$ and $t'_{-n} = t_n$, $n \geq 1$. The same kind of symmetry also appears in the variables t_n in Section 4.

6. RIEMANN HILBERT DATA

In the language of Takesaki and Takebe (see [27] and the references therein), to every solution of the dispersionless Toda hierarchy, one can associate a Riemann Hilbert data (or called the twistor data). Namely there exist two pairs of functions (r, h) and (\tilde{r}, \tilde{h}) of the variable p and t_0 such that

$$(6.1) \quad \begin{aligned} \{r, h\}_T &= r, & \{\tilde{r}, \tilde{h}\}_T &= \tilde{r}, \\ r(\mathcal{L}, \mathcal{M}) &= \tilde{r}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}), & h(\mathcal{L}, \mathcal{M}) &= \tilde{h}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}). \end{aligned}$$

Here \mathcal{M} and $\tilde{\mathcal{M}}$ are the Orlov-Schulman functions. They are defined so that they can be written as

$$(6.2) \quad \begin{aligned} \mathcal{M} &= \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + t_0 + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n}, \\ \tilde{\mathcal{M}} &= \sum_{n=1}^{\infty} -n t_{-n} \tilde{\mathcal{L}}^{-n} + t_0 + \sum_{n=1}^{\infty} -v_{-n} \tilde{\mathcal{L}}^n; \end{aligned}$$

and they form a canonical pair with \mathcal{L} and $\tilde{\mathcal{L}}$, i.e. $\{\mathcal{L}, \mathcal{M}\}_T = \mathcal{L}$ and $\{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\}_T = \tilde{\mathcal{L}}$. Conversely, Takasaki and Takebe also showed that if $(\mathcal{L}, \tilde{\mathcal{L}})$

are formal power series of the form (3.1), $\mathcal{M}, \tilde{\mathcal{M}}$ are formal functions of the form (6.2), and there exists (r, h) , (\tilde{r}, \tilde{h}) satisfying (6.1), then $(\mathcal{L}, \tilde{\mathcal{L}})$ is a solution to the dispersionless Toda hierarchy. The proof is formal. The main technique is comparing powers of p . However nothing is assumed about the convergence of the series. For the conformal welding problem we are studying, if we define the functions \mathcal{M} and $\tilde{\mathcal{M}}$ as (6.2), setting $\mathcal{L} = g$, $\tilde{\mathcal{L}} = f$, then from the definition of t_n and v_n , we obtain the following relations:

$$(6.3) \quad \mathcal{L}^{-1}\mathcal{M} = \tilde{\mathcal{L}}^{-1}, \quad \tilde{\mathcal{L}}\tilde{\mathcal{M}} = \mathcal{L},$$

on the unit circle. This is exactly the Riemann Hilbert problem studied by Takasaki [23] in connection to string theory. Here the superscript -1 on functions mean the reciprocal. Since $\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}$, we have the string equation

$$(6.4) \quad \{\mathcal{L}, \tilde{\mathcal{L}}^{-1}\} = \{\mathcal{L}, \mathcal{L}^{-1}\mathcal{M}\} = \mathcal{L}^{-1}\{\mathcal{L}, \mathcal{M}\} = 1.$$

The equations (6.3) are formally equivalent to

$$(6.5) \quad \mathcal{L}\mathcal{M}^{-1} = \tilde{\mathcal{L}}, \quad \mathcal{M} = \tilde{\mathcal{M}}.$$

Hence the associated Riemann Hilbert data can be written as $r(p, t_0) = pt_0^{-1}$, $h(p, t_0) = t_0$, $\tilde{r}(p, t_0) = p$ and $\tilde{h}(p, t_0) = t_0$. Since our f and g are C^1 functions and have absolutely convergent power series on the unit circle, the proof of Takasaki and Takebe can be adapted to our case to show that formally, (g, f) is a solution of the dispersionless Toda hierarchy, provided we are allowed to differentiate all power series term by term formally on S^1 . However, since t_n are not just formal variables, our proof has shown that everything is correct even analytically.

On the other hand, we see that the inverse functions $\mathcal{L} = g^{-1}$, $\tilde{\mathcal{L}} = f^{-1}$ and $\mathcal{M}, \tilde{\mathcal{M}}$ defined by (6.2) also satisfy the same relation (6.3), but on the curve \mathcal{C} . Hence by no means we can apply the proof of Takasaki and Takebe to conclude that (g^{-1}, f^{-1}) is a solution of the dispersionless Toda hierarchy, since the domain where their power series converge do not overlap in general. However, we have seen analytically that they indeed satisfy the dispersionless Toda hierarchy, and the proof is even simpler than the case of (g, f) .

Our interest in the inverse mappings (g^{-1}, f^{-1}) comes from the definition of the variables t_n . They are related to the Fourier coefficients of the homeomorphisms on the unit circle. However, these variables do not have a natural complex structure. In fact, $\text{Homeo}_C(S^1)$ is an 'odd' dimensional space since the homogenous space $S^1 \setminus \text{Homeo}_C(S^1)$ has a complex structure. We can choose a slice of $S^1 \setminus \text{Homeo}_C(S^1)$ in $\text{Homeo}_C(S^1)$ by imposing the condition that the coordinate t_0 is real. However, this condition will not be preserved under the t_n flows. On the other hand, t_{-n} in some sense is the

complex conjugate of t_n , since

$$\overline{\gamma^{-1}(w)} = \frac{1}{\gamma^{-1}(w^{-1})}.$$

We are going to see in the Appendix that in our choice of variables t_n , the subgroup of linear fractional transformations in $\text{Homeo}_C(S^1)$ correspond to the three dimensional subspace defined by $t_n = 0$ for all $n \geq 2$.

Here we will also like to compare our integrable structure on conformal welding with the integrable structure of conformal maps observed by Wiegmann and Zabrodin [29, 17, 20]. In the approach of Wiegman and Zabrodin, the coordinates t_n are defined as harmonic moments. Given a domain Ω^+ containing ∞ and bounded by an analytic curve \mathcal{C} , define

$$\begin{aligned} t_n &= \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-n} \bar{z} dz, & v_n &= \frac{1}{2\pi i} \oint_{\mathcal{C}} z^n \bar{z} dz, & n \geq 1; \\ t_0 &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \bar{z} dz, & v_0 &= \frac{1}{\pi} \iint_{\Omega^+} \log |z| d^2 z, \end{aligned}$$

and for $n \geq 1$, $t_{-n} = -\bar{t}_n$, $v_{-n} = -\bar{v}_n$. Let g be the Riemann mapping from the exterior disc to the domain Ω^+ normalized so that $g(\infty) = \infty$, $g'(\infty) > 0$. Let

$$(6.6) \quad \mathcal{L}(z) = g(z), \quad \text{and} \quad \tilde{\mathcal{L}}(z) = \frac{1}{\bar{g}(z^{-1})}.$$

Wiegmann and Zabrodin show that $(\mathcal{L}, \tilde{\mathcal{L}})$ is a solution to the dispersionless Toda hierarchy. Their solution also satisfies the relation (6.5). Now the close relations between the three solutions become apparent. In fact, from the relation (6.3), the dependence of v_m on the variables t_n are uniquely determined. Hence although the t_n and v_m have different interpretation in each of the approach, considering formally as functions of t_n , they in fact correspond to the same solution of the hierarchy. It is quite remarkable that we can define appropriate time variables on each of these problems, such that they are solutions of the same Riemann Hilbert problem. The Riemann Hilbert data (6.5) is significance for it implies the string equation (6.4). In fact, considering generalized inverse potential problem, Zabrodin has shown in [30] that there are other ways to define t_n and v_n so that (6.6) is a solution of the dispersionless Toda hierarchy. They correspond to different Riemann Hilbert data. Similarly, we can also choose other time variables so that the evolution of the interior and exterior mappings f and g and their inverses satisfy the dispersionless Toda hierarchy.

The solution of Wiegmann and Zabrodin has the advantage that it allows the introduction of a complex structure on the space of analytic curves. More precisely, on each slice with constant t_0 , $-t_{-n}$ is defined to be the complex conjugate of t_n . Moreover, as is observed by Takhtajan in [24], this gives a close analogy to conformal field theory. The tau function for analytic

curves is a partition function for the theory of free bosons on the space of analytic curves, and

$$\left\langle \frac{\partial}{\partial t_n}, \frac{\partial}{\partial t_m} \right\rangle = \frac{\partial^2 \log \tau}{\partial t_n \partial t_m}$$

defines a Hermitian metric on each constant t_0 slice. Another remarkable feature about the solution of Wiegmann and Zabrodin is that they are closely related to the Dirichlet boundary value problem (see [20]).

On the other hand, the solution of Wiegmann and Zabrodin cannot study the evolutions of the interior and exterior mappings using the same time variables. They have to introduce different time variables for the interior mapping problem and exterior mapping problem and these variables are related by a Legendre transformation (see [20]), which is similar to the symmetry of coordinates we discuss in Section 5.3. It will be interesting to investigate whether we can choose suitable time variables such that the solutions to the conformal welding problem also enjoy some of the good features of the solution of Wiegmann and Zabrodin. This is considered in the next section.

7. EXTENSION OF THE CONFORMAL MAPPING PROBLEM

In this section, we construct a theory that contains both our solution to the conformal welding problem and the solution of Wiegmann–Zabrodin to conformal mapping problem. We define the following spaces:

$$\begin{aligned} \mathfrak{S} &= \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \text{ univalent} \mid f(z) = a_1 z + a_2 z^2 + \dots; a_1 \neq 0; \right. \\ &\quad \left. f \text{ is extendable to a } C^1 \text{ homeomorphism of the plane.} \right\}, \\ \Sigma &= \left\{ g : \mathbb{D}^* \rightarrow \mathbb{C} \text{ univalent} \mid g(z) = bz + b_0 + b_1 z^{-1} + \dots; b \neq 0; \right. \\ &\quad \left. g \text{ is extendable to a } C^1 \text{ homeomorphism of the plane.} \right\}, \\ \mathfrak{D} &= \left\{ (f, g) \mid f \in \mathfrak{S}, g \in \Sigma; f'(0)g'(\infty) = a_1 b = 1. \right\}. \end{aligned}$$

We define the following functions on \mathfrak{D} :

$$\begin{aligned} t_n &= \frac{1}{2\pi i n} \oint_{S^1} \frac{(g(w))^{-n}}{f(w)} dg(w), \quad t_{-n} = \frac{-1}{2\pi i n} \oint_{S^1} g(w)(f(w))^{n-2} df(w), \\ v_n &= \frac{1}{2\pi i} \oint_{S^1} \frac{(g(w))^n}{f(w)} dg(w), \quad v_{-n} = \frac{-1}{2\pi i} \oint_{S^1} g(w)(f(w))^{-n-2} df(w), \quad n \geq 1, \\ t_0 &= \frac{1}{2\pi i} \oint_{S^1} \frac{1}{f(w)} dg(w) = \frac{1}{2\pi i} \oint_{S^1} \frac{g(w)}{(f(w))^2} df(w), \\ v_0 &= \frac{1}{2\pi i} \oint_{S^1} \left(\log \frac{g(w)}{w} \right) \frac{1}{f(w)} dg(w) - \frac{1}{2\pi i} \oint_{S^1} \left(\log \frac{f(w)}{w} \right) \frac{g(w)}{f(w)^2} df(w) \\ &\quad - \frac{1}{2\pi i} \oint_{S^1} \frac{g(w)}{f(w)} \frac{dw}{w}. \end{aligned}$$

As in Section 5, we can prove that $\{t_n, \bar{t}_n, n \in \mathbb{Z}\}$ is a complete set of local coordinates on \mathfrak{D} .

Let $\Omega_1^+ = f(\mathbb{D})$ and Ω_1^- its exterior. Let $\Omega_2^- = g(\mathbb{D}^*)$ and Ω_2^+ its exterior. We define the following functions:

$$S_{\pm}(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{(1/f)(w)g'(w)}{g(w) - z} dw, \quad z \in \Omega_2^{\pm},$$

$$\tilde{S}_{\pm}(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{g(w)f'(w)}{f(w)^2(f(w) - z)} dw, \quad z \in \Omega_1^{\pm}.$$

They are holomorphic functions in the respective domains. In a neighborhood of the origin or ∞ , they have the series expansion

$$S_+(z) = \sum_{n=1}^{\infty} n t_n z^{n-1}, \quad \tilde{S}_+(z) = - \sum_{n=1}^{\infty} v_{-n} z^{n-1},$$

$$S_-(z) = -t_0 z^{-1} - \sum_{n=1}^{\infty} v_n z^{-n-1}, \quad \tilde{S}_-(z) = -t_0 z^{-1} + \sum_{n=1}^{\infty} n t_{-n} z^{-n-1}.$$

Let F and G be the inverse functions of f and g respectively. Using Lemma 5.1, which still holds for $(f, g) \in \mathfrak{D}$, we can prove as in Propositions 5.3 and 5.5 that

Proposition 7.1. *Let $\kappa_{n,m}$ be the generalized Grunsky coefficients of the pair (F, G) . The variation of the function v_m with respect to the coordinates t_n , $n \in \mathbb{Z}$ is given by the following. For $m \neq 0$,*

$$\frac{\partial v_m}{\partial t_n} = -|mn| \kappa_{n,m}, \quad n \neq 0, \quad \frac{\partial v_m}{\partial t_0} = |m| \kappa_{0,m},$$

and for $m = 0$,

$$\frac{\partial v_0}{\partial t_n} = |n| \kappa_{n,0}, \quad n \neq 0, \quad \frac{\partial v_0}{\partial t_0} = 2 \log b,$$

and

$$\frac{\partial v_m}{\partial \bar{t}_n} = 0, \quad \text{for all } m, n.$$

Proof. We observe that

$$\begin{aligned}
\frac{\partial S_{\pm}}{\partial t}(z) &= \frac{1}{2\pi i} \oint_{S^1} \left(\frac{(\partial(1/f)/\partial t)(w)g'(w)}{g(w)-z} + \frac{(1/f)(w)(\partial^2 g(w)/\partial t \partial w)}{g(w)-z} \right. \\
&\quad \left. - \frac{(1/f)(w)g'(w)(\partial g/\partial t)(w)}{(g(w)-z)^2} \right) dw \\
&= \frac{1}{2\pi i} \oint_{S^1} \left(\frac{(\partial(1/f)/\partial t)(w)g'(w)}{g(w)-z} + \left(\frac{1}{f} \right)(w) \frac{\partial}{\partial w} \left(\frac{\partial g(w)/\partial t}{g(w)-z} \right) \right) dw \\
&= \frac{1}{2\pi i} \oint_{S^1} \left(\frac{(\partial(1/f)/\partial t)(w)g'(w)}{g(w)-z} - \frac{(1/f)'(w)(\partial g(w)/\partial t)}{g(w)-z} \right) dw \\
&= \frac{1}{2\pi i} \oint_{S^1} \frac{((\partial((1/f) \circ G))/\partial t)) \circ g(w)g'(w)}{g(w)-z} dw.
\end{aligned}$$

Similarly, we can show that

$$\frac{\partial \tilde{S}_{\pm}}{\partial t}(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{(\partial(g \circ F)/\partial t) \circ f(w)f'(w)}{(f(w))^2(f(w)-z)} dw.$$

The rest of the proof is the same. \square

We define the functions Φ and Ψ as in Section 5. Now we define the tau function τ to be the real valued function given by

$$\begin{aligned}
4 \log \tau &= \left(2t_0 v_0 - t_0^2 + \frac{1}{2\pi i} \oint_{S^1} \frac{g'(w)}{f(w)} (g(w)\Phi'(g(w)) + 2\Phi(g(w))) dw \right. \\
&\quad \left. + \frac{1}{2\pi i} \oint_{S^1} \frac{g(w)f'(w)}{(f(w))^2} (f(w)\Psi'(f(w)) - 2\Psi(f(w))) dw \right) \\
&\quad + \text{complex conjugate.}
\end{aligned}$$

Then we have

Proposition 7.2. *For all n , we have*

$$\frac{\partial \log \tau}{\partial t_n} = v_n, \quad \frac{\partial \log \tau}{\partial \bar{t}_n} = \bar{v}_n.$$

$\log \tau$ is a harmonic function on \mathfrak{D} . It generates two solutions of the dispersionless Toda hierarchy. Namely if we take t_n , $n \in \mathbb{Z}$ as the time variables, we find that $(\mathcal{L} = g, \tilde{\mathcal{L}} = f)$ is a solution to the dispersionless Toda hierarchy. Moreover, all the variations of the coefficients of f, g with respect to \bar{t}_n vanish. On the other hand, if we take \bar{t}_n as the time variables, then $(\mathcal{L} = \bar{g}, \tilde{\mathcal{L}} = \bar{f})$ is a solution to the dispersionless Toda hierarchy. We have introduced a complex structure on the space \mathfrak{D} such that the coefficients of f and g are holomorphic functions under this complex structure.

On the subspace of \mathfrak{D} defined by $f(z) = 1/\bar{g}(z^{-1})$, one finds that $b = g'(\infty)$ is real, $t_{-n} = -\bar{t}_n$, $v_{-n} = -\bar{v}_n$, for all $n \neq 0$, and t_0, v_0 are real. Hence $\{t_n, \bar{t}_n, n \geq 1, t_0\}$ form a set of local coordinates on this space, which is precisely the space considered by Wiegmann and Zabrodin. It is easy to

check that the definitions of t_n, v_n , $n \geq 0$ coincide with the definition of Wiegmann and Zabrodin. However, we do not say that all the evolution equations on the space \mathfrak{D} can be carried over to its subspace. The vector fields $\partial/\partial t_n$ have different meanings.

The subspace of \mathfrak{D} defined by $f(S^1) = g(S^1)$ corresponds to the conformal welding problem we discuss in Section 5. Hence we see that the solution of Wiegmann and Zabrodin and our solution to the conformal welding problem actually correspond to two different subsystem of the same system.

Acknowledgement The author would like to thank the Ministry of Science, Technology and Innovation of Malaysia for funding this project under eScienceFund 06-02-01-SF0021.

APPENDIX A. THE SUBGROUP OF LINEAR FRACTIONAL TRANSFORMATIONS

Here we consider the simplest case of domains, namely discs. The most general conformal maps f and g mapping \mathbb{D} and \mathbb{D}^* to the interior and exterior of a circle and satisfying our normalization conditions are given by

$$f(z) = \frac{1}{b} \frac{z}{1+az}, \quad |a| < 1, \quad g(z) = bz + c, \quad b \neq 0.$$

The necessary condition that f and g map S^1 to the same circle is that $\gamma = g^{-1} \circ f$ is a linear fractional transformation on the unit circle, i.e. it belongs to $\text{PSL}(2, \mathbb{R})$. This implies that

$$b = \frac{e^{\frac{i\alpha}{2}}}{\sqrt{(1-|a|^2)}}, \quad c = -\frac{\bar{a}e^{-\frac{i\alpha}{2}}}{\sqrt{(1-|a|^2)}}, \quad \text{and} \quad \gamma(z) = e^{-i\alpha} \frac{z + \bar{a}}{1 + az}.$$

For the inverse mappings (g^{-1}, f^{-1}) , we consider the Fourier expansions:

$$\begin{aligned} \gamma(w) &= \bar{a}e^{-i\alpha} + \sum_{n=0}^{\infty} (-1)^n e^{-i\alpha} a^n (1-|a|^2) w^{n+1}, \\ \frac{1}{\gamma^{-1}}(w) &= -a + \sum_{n=0}^{\infty} e^{-i(n+1)\alpha} (1-|a|^2) \bar{a}^n w^{-n-1}. \end{aligned}$$

The variables t_n and v_n are given by

$$\begin{aligned} t_1 &= -a, \quad t_0 = e^{-i\alpha}(1-|a|^2), \quad t_{-1} = -\bar{a}e^{-i\alpha}, \quad t_n = 0, \quad \forall |n| \geq 2, \\ v_n &= e^{-i(n+1)\alpha}(1-|a|^2)\bar{a}^n, \quad v_{-n} = (-1)^{n-1} e^{-i\alpha} a^n (1-|a|^2), \quad n \geq 1. \end{aligned}$$

Hence

$$v_n = (-1)^n t_0 t_{-1}^n, \quad v_{-n} = -t_0 t_1^n, \quad v_0 = t_0 \log t_0 - t_0 - t_1 t_{-1},$$

and the tau function is given by

$$\log \tau = \frac{t_0^2}{2} \log t_0 - \frac{3}{4} t_0^2 - t_0 t_1 t_{-1}.$$

The restriction b being real amount to restricting $\alpha = 0$. In that case t_0 depend on t_1 and t_{-1} by $t_0 = 1 - t_1 t_{-1}$.

For the mappings (g, f) ,

$$t_n = \frac{1}{2\pi i n} \oint_{S^1} \frac{g(w)^{-n}}{f(w)} dg(w) = \begin{cases} ab = \frac{ae^{\frac{i\alpha}{2}}}{\sqrt{1-|a|^2}}, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

$$t_{-n} = \frac{-1}{2\pi i n} \oint_{S^1} f(w)^{n-2} g(w) df(w) = \begin{cases} -c = \frac{\bar{a}e^{-\frac{i\alpha}{2}}}{\sqrt{1-|a|^2}} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

Similarly, we find that

$$\begin{aligned} t_0 &= b^2 = e^{i\alpha}(1 - |a|^2)^{-1}, \\ v_n &= b^2 c^n, \quad v_{-n} = -a^n b^{n+2}, \quad n \geq 1. \end{aligned}$$

Hence, as functions of t_n , we still have

$$v_n = (-1)^n t_0 t_{-1}^n, \quad v_{-1} = -t_0 t_1^n, \quad v_0 = t_0 \log t_0 - t_0 - t_1 t_{-1},$$

and the tau function is given by

$$\log \tau = \frac{t_0^2}{2} \log t_0 - \frac{3}{4} t_0^2 - t_0 t_1 t_{-1}.$$

REFERENCES

- [1] L. M. Alonso, *Genus-zero Whitham hierarchies in conformal-map dynamics*, Phys. Lett. B. **641** (2006), 466–473.
- [2] L. M. Alonso and E. Medina, *Solutions of the dispersionless Toda hierarchy constrained by string equations*, J. Phys. A: Math. Gen. **37** (2004), 12005–12017.
- [3] L. M. Alonso and E. Medina, *Exact solutions of integrable 2D, contour dynamics*, Phys. Lett. B **610** (2005), 277–282.
- [4] L. M. Alonso, E. Medina and M. Manas, *String equations in Whitham hierarchies: tau-functions and Virasoro constraints*, J. Math. Phys. **47** (2006), 083512.
- [5] M. Bauer and D. Bernard, *2D growth processes: SLE and Loewner chains*, Phys. Rep. **432** (2006), 115–221.
- [6] A. Boyarsky, A. Marshakov, O. Ruchayskiy, P. Wiegmann and A. Zabrodin, *Associativity equations in dispersionless integrable hierarchies*, Phys. Lett. B **515** (2001), 483–492.
- [7] D. Crowdy, *The Benney hierarchy and the Dirichlet boundary problem in two dimensions*, Phys. Lett. A **343** (2005), 319–329.
- [8] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 259, Springer-Verlag, New York, 1983.
- [9] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs, vol. 76, American Mathematical Society, Providence, RI, 2000.
- [10] F. P. Gardiner and D. P. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), no. 4, 683–736.
- [11] A. A. Kirillov, *Kähler structure on the K-orbits of a group of diffeomorphisms of the circle*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 42–45.

- [12] A. A. Kirillov and D. V. Yuriev, *Kähler geometry of the infinite-dimensional homogeneous space $M = \text{diff}_+(S^1)/\text{rot}(S^1)$* , Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 35–46, 96.
- [13] B. Konopelchenko, L. M. Alonso and O. Ragnisco, *The partial derivative-approach to the dispersionless KP hierarchy*, J. Phys. A: Math. Gen. **34** (2001), 10209–10217.
- [14] B. Konopelchenko and L. M. Alonso, *Dispersionless scalar integrable hierarchies, Whitham hierarchy, and the quasiclassical $\bar{\partial}$ -dressing method*, J. Math. Phys. **43** (2002), 3807–3823.
- [15] B. Konopelchenko and L. M. Alonso, *Nonlinear dynamics on the plane and integrable hierarchies of infinitesimal deformations*, Stud. Appl. Math. **109** (2002), 313–336.
- [16] I. K. Kostov, *String equation for string theory on a circle*, Nucl. Phys. B **624** (2002), 146–162.
- [17] I. K. Kostov, I. M. Krichever, M. Mineev-Weinstein, A. Zabrodin, and P. B. Wiegmann, *The τ -function for analytic curves*, Random matrix models and their applications, Math. Sci. Res. Inst. Publ., vol. 40, Cambridge Univ. Press, Cambridge, 2001, pp. 285–299.
- [18] I. Krichever, A. Marshakov and A. Zabrodin, *Integrable structure of the Dirichlet boundary problem in multiply-connected domains*, Comm. Math. Phys. **259** (2005), 1–44.
- [19] O. Lehto, *Univalent functions and Teichmüller spaces*, Graduate Texts in Mathematics, vol. 109, Springer-Verlag, New York, 1987.
- [20] A. Marshakov, P. Wiegmann and A. Zabrodin, *Integrable structure of the Dirichlet boundary problem in two dimensions*, Comm. Math. Phys. **227** (2002), no. 1, 131–153.
- [21] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975, With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.
- [22] D. Prokhorov and A. Vasil'ev, *Univalent functions and integrable systems*, Comm. Math. Phys. **262** (2006), 393–410.
- [23] K. Takasaki, *Dispersionless Toda hierarchy and two-dimensional string theory*, Comm. Math. Phys. **170** (1995), no. 1, 101–116.
- [24] L. A. Takhtajan, *Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. Current correlation functions*, Lett. Math. Phys. **56** (2001), no. 3, 181–228, EuroConférence Moshé Flato 2000, Part III (Dijon).
- [25] L. P. Teo, *Analytic functions and integrable hierarchies—characterization of tau functions*, Lett. Math. Phys. **64** (2003), no. 1, 75–92.
- [26] L. P. Teo, *The Velling-Kirillov metric on the universal Teichmüller curve*, J. Anal. Math. **93** (2004), 271–307.
- [27] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. **7** (1995), no. 5, 743–808.
- [28] L. A. Takhtajan and L. P. Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc., **183** (2006), no. 861, vi+119.
- [29] P. B. Wiegmann and A. Zabrodin, *Conformal maps and integrable hierarchies*, Comm. Math. Phys. **213** (2000), no. 3, 523–538.
- [30] A. V. Zabrodin, *The dispersionless limit of the Hirota equations in some problems of complex analysis*, Teoret. Mat. Fiz. **129** (2001), no. 2, 239–257.
- [31] A. V. Zabrodin, *Growth processes related to the dispersionless Lax equations*, Physica D **235** (2007), 101–108.